International Journal of Theoretical Physics, Vol. 45, No. 3, March 2006 (© 2006)
DOI: 10.1007/s10773-006-9048-9

Conservation Laws and Exact Solutions for some Nonlinear Partial Differential Equations

A. H. Khater,[1,2,3] D. K. Callebaut,[2] and S. M. Sayed[1]

Received June 30, 2005; accepted January 26, 2006
Published Online: April 12, 2006

An effective algorithmic method (Anco, S. C. and Bluman, G. (1996). *Journal of Mathematical Physics* **37**, 2361; Anco, S. C. and Bluman, G. (1997). *Physical Review Letters* **78**, 2869; Anco, S. C. and Bluman, G. (1998). *European Journal of Applied Mathematics* **9**, 254; Anco, S. C. and Bluman, G. (2001). *European Journal of Applied Mathematics* **13**, 547; Anco, S. C. and Bluman, G. (2002). *European Journal of Applied Mathematics* **13**, 567) is used for finding the local conservation laws for some nonlinear partial differential equations. The method does not require the use or existence of a variational principle and reduces the calculation of conservation laws to solving a system of linear determining equations similar to that of finding symmetries. An explicit construction formula is derived which yields a conservation law for each solution of the determining system. Different methods to construct new exact solution classes for the same nonlinear partial differential equations are also presented, which are named hyperbolic function method and the Bäcklund transformations. On the other hand, other methods and transformations are developed to obtain exact solutions for some nonlinear partial differential equations.

KEY WORDS: conservation laws; nonlinear partial differential equations; traveling wave solutions.

PACS Numbers: 02.30.Jr - Partial differential equations, 11.30.-j - Conservation laws, 03.50.-z -Classical field theory.

1. INTRODUCTION

In the study of differential equations (DEs), conservation laws have many significant uses, particularly with regard to integrability and linearization, constants of motion, analysis of solutions, and numerical solution methods. Consequently, an important problem is how to calculate all of the conservation laws for given DEs.

[1] Department of Mathematics, Faculty of Science, Cairo University, Beni-Suef, Egypt.
[2] Departement Natuurkunde, CDE, University of Antwerp (UA), B-2610 Antwerp, Belgium.
[3] To whom correspondence should be addressed at Department of Mathematics, Faculty of Science, Cairo University, Beni-Suef, Egypt; e-mail: Khater_ah@yahoo.com.

For a DE with a variational principle, Noether's theorem (Noether, 1918; Bessel-Hagen, 1921; Olver, 1977, 1986; Tonti, 1967, 1968a,b,c, 1984; Khater *et al.*, 1997, 1999, 2001) gives a formula for obtaining the local conservation laws by use of symmetries of the action. One usually attempts to find these symmetries by noting that any symmetry of the action leaves the extremals of the action invariant and hence gives rise to a symmetry of the DE. However, all symmetries of a DE do not necessarily arise from symmetries of the action when there is a variational principle. For example, if a DE is scaling invariant, then the action is often not invariant. Indeed, it is often computationally awkward to determine the symmetries of the action and carry out the calculation with the formula to obtain a conservation law.

Moreover, in general, a DE need not have a variational principle even allowing for a change of variables. Therefore, it is more effective to seek a direct, algorithmic method without involving an action principle to find the conservation laws of a given DE.

Anco and Bluman (1996, 1997) introduced an algorithmic approach replacing Noether's theorem so as to allow one to obtain all local conservation laws for any DE whether or not it has a variational principle. Details of this approach for the situation of ordinary differential equations (ODEs) are given by Anco and Bluman (1998). Here, we concentrate on the situation of nonlinear partial differential equations (NLPDEs).

In the case of partial differential equation (PDE) with a variational principle, the approach shows how to use the symmetries of the (NLPDEs) to directly construct the conservation laws. The symmetries of a NLPDE satisfy a linear determining equation, for which there is a standard algorithmic method (Mikhailov, 1981; Ablowitz and Clarkson, 1991; Anco and Bluman, 2001, 2002; Wolf, 2002; Khater *et al.*, 2004a) to seek all solutions. There is also an invariance condition, involving just the NLPDE and its symmetries, which is necessary and sufficient for a symmetry of a NLPDE with a variational principle to correspond to a symmetry of the action. The invariance condition can be checked by an algorithmic calculation and, in addition, leads to a direct construction formula for a conservation law in terms of the symmetry and the NLPDE. This approach makes no use of the variational principle for the NLPDE (Bluman and Kumi, 1989; Bluman, 1990, 1993).

In the case of a NLPDE without a variational principle, the approach involves replacing symmetries by adjoint symmetries of the NLPDE. The adjoint symmetries satisfy a linear determining equation that is the adjoint of the determining equation for symmetries. Geometrically, symmetries of a NLPDE describe motions on the solution space of the NLPDE. Adjoint symmetries in general do not have such an interpretation (Bluman and Kumi, 1987, 1988; Bluman *et al.*, 1988). The invariance condition on symmetries is replaced by an adjoint invariance condition on adjoint symmetries and there is a corresponding direct construction formula for obtaining the conservation laws in terms of the adjoint symmetries and the NLPDE. The adjoint invariance condition is a necessary and sufficient determining condition for an adjoint symmetry to yield a conservation law. This paper uses an algorithm to obtain all local conservation laws for any system of field equations. The algorithm uses a formula, which directly generates the conservation laws and does not depend on the system having a Lagrangian formulation, in contrast to Noerther's theorem, which requires a Lagrangian. Example is considered including dissipative systems inherently having no Lagrangian.

Finding conservation laws is important in the study of physical systems. Given a system of field equations arising from a Lagrangian, one can obtain all local conservation laws of the system algorithmically by using Noether's theorem. The algorithm involves the following steps: (1) find all local symmetries of the given Lagrangian system (Olver, 1986; Bluman and Kumi, 1989); (2) check, which of the symmetries leave the Lagrangian invariant to within a local divergence; and (3) for each such symmetry construct a conservation law through the variational relation between the Lagrangian and the field equations (Olver, 1986; Bluman and Kumi, 1989). However, given a system without a Lagrangian formulation, one currently lacks a corresponding algorithm to find local conservation laws of the system.

Moreover, we seeking the exact solutions of the NLPDEs play an important role in the nonlinear problems, a number of methods have been developed, such as inverse scattering theory (Gardner *et al.*, 1967), Hirota's bilinear methods (Hirota, 1973), the truncated Painlevé expansion (Weiss *et al.*, 1983; Cariello and Tabor, 1991), homogeneous balance method (Wang, 1995, 1996), reduction of the NLPDE to a quadrature problem (Otwinowski *et al.*, 1988), etc.

The main aim of this paper is to derive conservation laws and new classes of exact traveling wave solutions for some NLPDEs.

This paper is organized as follows. In Section 2 we summarize the effective algorithmic method for finding the local conservation laws for some NLPDEs. We also illustrate the use of the direct conservation laws method on some NLPDEs examples. In Section 3 we find new classes of exact traveling wave solutions for some NLPDEs. Finally, we give some conclusions in Section 4.

2. CONSERVATION LAWS FOR SOME NLPDEs

This section illustrates an algorithm to obtain all local conservation laws for any system of field equations whether or not the system has a Lagrangian formulation. The algorithm uses an adjoint invariance condition together with a formula, which directly yields a conservation law for any solution of the condition, and involves the following steps: (1) linearize the given system of field equations and find the adjoint system of the linearized system; (2) find all solutions of the adjoint system; (3) check which of the solutions satisfy our adjoint invariance condition; and (4) for each such solution construct a conservation law directly by the formula defined in (Anco and Bluman, 1998). The linearized system and the adjoint system are the same if and only if the given system is self-adjoint, in which case the system has a Lagrangian formulation. For a Lagrangian system, the solutions of the linearized system are simply the symmetries of the field equations, and the adjoint invariance condition is equivalent to the condition that symmetry leaves the Lagrangian invariant to, within a local divergence. However, the construction of conservation laws for such a system is considerably simpler using this algorithm than using Noether's theorem since the adjoint invariance condition and conservation law formula do not require the expression for the local divergence arising from invariance of the Lagrangian which is a cumbersome yet essential step in Noether's theorem (Khater *et al.*, 2001b,c, 2002a,b, 2003, 2004c). In this algorithm, for any given system of the field equations, the formula used for the construction of conservation laws is well defined for an arbitrary solution of the adjoint system. Consequently, after steps (1) and (2), one can bypass step (3) and simply insert each solution of the adjoint system into the conservation law formula, then check whether the resulting expression is conserved for all solutions of the field equations. Consider any given system of field equations (Anco and Bluman, 1997)

$$G_\Omega[u] = 0, \tag{1}$$

for field variables $u^\sigma(x^i)$, with time and space coordinates represented by variables x^i. The linearization of system (1) is given by

$$\ell_{\Omega_\rho}[u]\eta^\rho = 0, \tag{2}$$

where

$$\ell_{\Omega_\rho}[u] = G_{\Omega_\rho}[u] + G^i_{\Omega_\rho}[u]D_i + G^{ij}_{\Omega_\rho}[u]D_i D_j + \cdots \tag{3}$$

where D_i denote the coordinate derivative operator with respect to x^i, and $G_{\Omega_\rho}[u] = \partial G_\Omega[u]/\partial u^\rho$, $G^i_{\Omega_\rho}[u] = \partial G_\Omega[u]/\partial D_i u^\rho$, etc. The adjoint system of Eq. (2) is given by

$$\ell^*_{{}_\rho\Omega}[u]\Lambda^\Omega = G_{\Omega_\rho}[u]\Lambda^\Omega + D_i\big(G^i_{\Omega_\rho}[u]\Lambda^\Omega\big) + D_i D_j\big(G^{ij}_{\Omega_\rho}[u]\Lambda^\Omega\big) + \cdots = 0, \tag{4}$$

with $\ell^*_{{}_\rho\Omega}[u]$ defining the formal adjoint of $\ell_{{}_\rho\Omega}[u]$. In (2) and (4), u^σ is an arbitrary solution of the field equations as shown in (1), while η^σ and Λ^Ω are variables depending locally on u^σ and derivatives of u^σ. The adjoint invariance condition on a solution $\Lambda^\Omega[u]$ of (4) is given by

$$\begin{aligned}\ell^*_{{}_\rho\Omega}[U]\Lambda^\Omega[U] = {} & -\Lambda^\Omega_\sigma[U]G_\Omega[U] + D_i\big(\Lambda^{\Omega_i}_\sigma[U]G_\Omega[U]\big)\\ & - D_i D_j\big(\Lambda^{\Omega_{ij}}_\sigma[U]G_\Omega[U]\big) + \cdots\end{aligned} \tag{5}$$

for arbitrary $U^\sigma(x^i) = u^\sigma(x^i)$ which are field variables not subject to the field equations, where $\Lambda^\Omega_\sigma[U] = \partial\Lambda^\Omega[U]/\partial U^\sigma$, $\Lambda^{\Omega_i}_\sigma[U] = \partial\Lambda^\Omega_\sigma[U]/\partial D_i U^\sigma$, etc. Given any $\Lambda^\Omega[U]$ satisfy condition (5), we have the following conservation law on all solutions u^σ of the field equations as shown in (1):

$$D_i \Phi^i[u] = 0, \tag{6}$$

where (Miura *et al.*, 1968; Whitham, 1974)

$$\Phi^i[u] = \int_0^1 \frac{d\lambda}{\lambda}(S^i[U] + N^i_\sigma[U]U^\sigma + N^{ij}_\sigma[U]D_jU^\sigma + \cdots)|U = \lambda u \tag{7}$$

$$S^i[U] = \Lambda^\Omega[U]G^i_{\Omega_\sigma}[U]U^\sigma + \left(\Lambda^\Omega[U]G^{ij}_{\Omega_\sigma}[U]D^jU^\sigma\right)$$
$$-D_j\left(\Lambda^\Omega[U]G^{ij}_{\Omega_\sigma}[U]U^\sigma\right) + \cdots, \tag{8}$$

$$N^i_\sigma[U] = \Lambda^{\Omega_i}[U]G_\Omega[U] - D_j\left(\Lambda^{\Omega_{ij}}_\sigma[U]G_\Omega[U]\right) + \cdots, \tag{9}$$

$$N^{ij}_\sigma[U] = \Lambda^{\Omega_{ij}}[U]G_\Omega[U] - D_k(\Lambda^{\Omega_{ijk}}_\sigma[U]G_\Omega[U]) + \cdots. \tag{10}$$

In Olver (1986) Bluman and Kumi 1989) the proof that the expression $\Phi^i[u]$ satisfies Eq. (6), we also show that for any given system as shown in (1) all local conservation laws are obtained by this algorithm.

To illustrate the adjoint invariance condition in Eq. (5) and conservation law formula an in Eq. (7), we now consider several examples of scalar field equations in two space-time dimensions. For the sequel, $x^0 = t,\ x^1 = x$ denote time and space coordinates, $D_0 = D_t,\ D_1 = D_x$ denote coordinate derivatives, and $u(x, t)$ denotes a scalar field.

This section ends with the examples of conservation laws for some NLPDEs.

2.1. Compound Korteweg-de-Vries (KdV)–Burgers' Equation

We consider the compound KdV–Burgers' equation

$$G(u) = D_tu + qu^2D_xu + puD_xu + rD_x^2u - sD_x^3u = 0, \tag{11}$$

where p, q, r, s are constants. This field equation lacks a direct Lagrangian formulation. Linearizing (11) leads to the operator

$$\ell[u] = D_t + (pu + qu^2)D_x + (p + 2qu)D_xu + rD_x^2 - sD_x^3, \tag{12}$$

and taking the adjoint yields

$$\ell^*[u] = -D_t - (pu + qu^2)D_x + rD_x^2 + sD_x^3 \neq \ell[u]. \tag{13}$$

This leads to the adjoint system of the linearized compound KdV–Burgers' equation

$$\ell^*[u]\Lambda = -D_t\Lambda - (pu + qu^2)D_x\Lambda + rD_x^2\Lambda + sD_x^3\Lambda = 0, \tag{14}$$

where Λ is a scalar field. System (14) has solutions $\Lambda = f(x, t)$ given by

$$-D_tf - (pu + qu^2)D_xf + rD_x^2f + sD_x^3f = 0, \tag{15}$$

with no dependence on u. These solutions trivially satisfy the adjoint invariance condition as shown in (5) since $\ell^*[U]\Lambda$ and $\partial\Lambda/\partial U$, $\partial\Lambda/\partial D_t U$, $\partial\Lambda/\partial D_x U$, etc., identically vanish for an arbitrary $U(x,t)$. There are no solutions, which have explicit local dependence on u or derivatives of u. Evaluating the conservation law formula (7) for $\Lambda = f(x,t)$, we obtain the conserved density

$$\Phi^i[u] = \int_0^1 \frac{d\lambda}{\lambda} \Lambda[U]U|_{U=\lambda u} = \int_0^1 f u d\lambda = f u, \tag{16}$$

where $f(x,t)$ is an arbitrary solution of the adjoint compound KdV–Burgers' equation (15). This yield the infinite number of elementary local conserved densities of the compound KdV–Burgers' equation (11).

The compound KdV–Burgers' equation (11) can be thought of as a generalization of the KdV, mKdV and Burgers' equations, involving nonlinear dispersion and dissipation effects. As particular cases,

(1) the compound KdV equation

$$D_t u + p u D_x u + q u^2 D_x u - s D_x^3 u = 0, \quad \text{when } r = 0, \text{ and } pqs \neq 0, \tag{17}$$

(2) the KdV–Burgers' equation

$$D_t u + r D_x^2 u + q u^2 D_x u - s D_x^3 u = 0, \text{ when } p = 0 \text{ and } qrs \neq 0, \tag{18}$$

(3) the mKdV equation

$$D_t u + q u^2 D_x u - s D_x^3 u = 0, \quad \text{when } r = 0, \ p = 0. \tag{19}$$

2.2. Liouville's Equation

Consider Liouville's equation in physical form

$$G[u] = D_t^2 u + D_x^2 u - e^u = 0, \tag{20}$$

then a Lagrangian $L[u]$ is given by

$$L[u] = \frac{1}{2}(D_t u)^2 + \frac{1}{2}(D_x u)^2 - e^u. \tag{21}$$

Linearizing (20) leads to the operator

$$\ell[u] = D_t^2 + D_x^2 - e^u, \tag{22}$$

which is self-adjoint, $\ell[u] = \ell^*[u]$. Thus, the adjoint system of the linearized field equation is given by

$$\ell^*[u]\Lambda = D_t^2\Lambda + D_x^2\Lambda - e^u\Lambda = 0, \tag{23}$$

where Λ is a scalar field which depends on locally on u, with u satisfying (20). System (23) is simply the determining equation for the local symmetries $\delta u = \Lambda[u]$ of the field equation (20). The solutions of the system are given by a time translation $\Lambda[u] = D_t u$, a space translation $\Lambda[u] = D_x u$, and a scaling $\Lambda[u] = tD_t u + xD_x u + 2$. Through our adjoint invariance condition and conservation law formula, we now show that the time and the scaling translation leads to the expected energy conservation laws. First, for $\Lambda = D_t u$, we readily see $\ell^*[U]\Lambda[U] = D_t(D_t^2 U + D_x^2 U - e^U) = D_t G[U]$, for any arbitrary $U(x,t)$ since

$$\partial\Lambda[U]/\partial U = 0,\ \partial\Lambda[U]/\partial D_t U = 1,\ \partial\Lambda[U]/\partial D_x U = 0, \tag{24}$$

etc., we have

$$\begin{aligned}&-\partial\Lambda[U]/\partial U(G[U]) + D_t(\partial\Lambda[U]/\partial D_t U(G[U]))\\ &\quad + D_x(\partial\Lambda[U]/\partial D_x U(G[U])) + \cdots\\ &\quad = D_t(\partial\Lambda[U]/\partial D_t U(G[U])) = D_t G[U].\end{aligned} \tag{25}$$

Thus $\Lambda[u] = D_t u$ satisfies the adjoint invariance condition (5). However, for $\Lambda[u] = tD_t u + xD_x u + 2$, we find

$$\begin{aligned}\ell^*[U]\Lambda[U] &= tD_t\big(D_t^2 U + D_x^2 U - e^U\big) + 2\big(D_t^2 U + D_x^2 U - e^U\big)\\ &+xD_x\big(D_t^2 U + D_x^2 U - e^U\big) = tD_t G[U] + 2G[U] + xD_x G[U],\end{aligned} \tag{26}$$

and

$$\begin{aligned}&-\partial\Lambda[U]/\partial U(G[U]) + D_t(\partial\Lambda[U]/\partial D_t U(G[U]))\\ &\quad + D_x(\partial\Lambda[U]/\partial D_x U(G[U])) + \cdots = tD_t G[U] + 2G[U] + xD_x(G[U])).\end{aligned} \tag{27}$$

Using $\partial\Lambda[U]/\partial U = 0,\ \partial\Lambda[U]/\partial D_t U = t,\ \partial\Lambda[U]/\partial D_x U = x$, etc., we see that the adjoint invariance condition (5) is satisfied. For any of the solutions $\Lambda[u]$ satisfying the adjoint invariance condition, the conservation law formula (7) leads to the conserved density

$$\Phi^0[u] = \int_0^1 d\lambda(\Lambda[\lambda u]D_t u - uD_t[\lambda u] + u\frac{1}{\lambda}\frac{\partial\Lambda[\lambda u]}{\partial D_t u}G[\lambda u]). \tag{28}$$

Evaluating (28) for $\Lambda[u] = D_t u$, then

$$\Phi^0[u] = \int_0^1 d\lambda[\lambda(D_t u)^2 + \lambda u(D_x u)^2 - ue^{\lambda u}] = \frac{1}{2}\big(D_t u\big)^2 + \frac{1}{2}u(D_x^2 u) - e^u + 1,$$

then, it reads

$$\Phi^0[u] = \frac{1}{2}(D_t u)^2 - \frac{1}{2}(D_x u)^2 + D_x\big(\frac{1}{2}uD_x u\big) - e^u + 1. \tag{29}$$

For $\Lambda[u] = tD_t u + xD_x u + 2$, we get

$$\Phi^0[u] = \int_0^1 d\lambda[\lambda t\big(D_t u\big)^2 + \lambda t u(D_x^2 u) + \lambda x(D_t u)(D_x u) + \lambda u(D_t u) - \lambda x u D_{tx} u - u t e^{\lambda u}],$$

$$\Phi^0[u] = \frac{1}{2} t[(D_t u)^2 + u(D_x u)^2] + t(1 - e^u) + \frac{1}{2} x[(D_x u)(D_t u) - u(D_{tx} u)] + \frac{1}{2}(4 - u)D_t u,$$

we obtain

$$\Phi^0[u] = t\left[\frac{1}{2}(D_t u)^2 - \frac{1}{2}(D_x u)^2 - e^u + D_x(\frac{1}{2} u D_x u) + 1\right] + \frac{1}{2}(4 - u)D_t u + \frac{1}{2} x[(D_x u)(D_t u) - u(D_{tx} u)]. \quad (30)$$

Equations (29) and (30) are local conserved densities, up to addition of trivial conserved densities.

2.3. Sine Gordon (SG) Equation

Consider a Lagrangian

$$L[u] = \frac{1}{2} u_x u_t + \cos u. \quad (31)$$

The field equation from $L[u]$ is given by

$$G[u] = D_{xt} u - \sin u = 0, \quad (32)$$

is called SG equation. Linearizing (32) leads to the operator

$$\ell[u] = D_{xt} - \cos u, \quad (33)$$

which is self-adjoint, $\ell[u] = \ell^*[u]$. Thus, the adjoint system of the linearized field equation is given by

$$\ell^*[u]\Lambda = D_{xt}\Lambda - (\cos u)\Lambda = 0. \quad (34)$$

The solutions of the system are given by a time translation $\Lambda[u] = D_t u$, a space translation $\Lambda[u] = D_x u$, and a scaling $\Lambda[u] = tD_t u + xD_x u + 2$. Through our adjoint invariance condition and conservation law formula, we now show that the time and the scaling translation leads to the expected energy conservation laws. First, for $\Lambda = D_t u$, we readily see $\ell^*[U]\Lambda[U] = D_t(D_{xt} U - \sin U) = D_t G[U]$, for any arbitrary $U(x, t)$. Since $\partial\Lambda[U]/\partial U = 0$, $\partial\Lambda[U]/\partial D_t U =$

1, $\partial\Lambda[U]/\partial D_x U = 0$, etc., we have

$$
\begin{aligned}
&-\partial\Lambda[U]/\partial U(G[U]) + D_t(\partial\Lambda[U]/\partial D_t U(G[U])) \\
&+ D_x(\partial\Lambda[U]/\partial D_x U(G[U])) + \cdots \\
&= D_t(\partial\Lambda[U]/\partial D_t U(G[U])) = D_t G[U]. \qquad (35)
\end{aligned}
$$

Thus, $\Lambda[u] = D_t u$ satisfies the adjoint invariance condition as shown in (5). However, for $\Lambda[u] = tD_t u + xD_x u + 2$, we find

$$
\begin{aligned}
\ell^*[U]\Lambda[U] &= tD_t(D_{xt}U - \sin U) + 2(D_{xt}U - \sin U) + xD_x(D_{xt}U - \sin U) \\
&= tD_t G[U] + 2G[U] + xD_x G[U], \qquad (36)
\end{aligned}
$$

and

$$
\begin{aligned}
&-\partial\Lambda[U]/\partial U(G[U]) + D_t(\partial\Lambda[U]/\partial D_t U(G[U])) \\
&+ D_x(\partial\Lambda[U]/\partial D_x U(G[U])) + \cdots = tD_t G[U] + 2G[U] + xD_x(G[U])). \qquad (37)
\end{aligned}
$$

Using $\partial\Lambda[U]/\partial U = 0$, $\partial\Lambda[U]/\partial D_t U = t$, $\partial\Lambda[U]/\partial D_x U = x$, etc., we see that the adjoint invariance condition (5) is satisfied. For any of the solutions $\Lambda[u]$ satisfying the adjoint invariance condition, the conservation law formula (7) leads to the conserved density

$$
\Phi^0[u] = \int_0^1 d\lambda(\Lambda[\lambda u]D_t u - uD_t[\lambda u] + u\frac{1}{\lambda}\frac{\partial\Lambda[\lambda u]}{\partial D_t u}G[\lambda u]). \qquad (38)
$$

Evaluating (38) for $\Lambda[u] = D_t u$, then

$$
\Phi^0[u] = \int_0^1 d\lambda[\lambda(D_t u)^2 + \lambda u(D_{xt}u) - \lambda u D_t^2 u - u\sin\lambda u],
$$

we get

$$
\Phi^0[u] = (D_t u)^2 - D_t\left(\frac{1}{2}uD_t u\right) + \frac{1}{2}uD_{xt}u + \cos\, u - 1. \qquad (39)
$$

For $\Lambda[u] = tD_t u + xD_x u + 2$, we obtain

$$
\begin{aligned}
\Phi^0[u] = \int_0^1 d\lambda[&\lambda t(D_t u)^2 + \lambda tu(D_{xt}u) + \lambda x(D_t u)(D_x u) - \lambda u(D_t u) \\
&-\lambda u D_t^2 u - \lambda xuD_{tx}u + 2D_{tx}u - tu\sin\, u],
\end{aligned}
$$

then, it reads

$$
\begin{aligned}
\Phi^0[u] \ &= \frac{1}{2}(D_t u)[tD_t u + xD_x u - uD_t u - u + 4] \\
&+\frac{1}{2}(t - x)uD_{xt}u + t(\cos u - 1). \qquad (40)
\end{aligned}
$$

Equations (39) and (40) are local conserved densities, up to addition of trivial conserved densities.

2.4. The Camassa–Holm (CH) Equation

Consider the CH equation

$$G[u] = D_t u - D_{xxt} u + 3u D_x u - 2D_x u D_x^2 u - u D_x^3 u = 0. \tag{41}$$

This field equation lacks a direct Lagrangian formulation. Linearizing (41) leads to the operator

$$\ell[u] = D_t - D_{xxt} + 3u D_x + 3D_x u - 2D_x^2 u D_x - 2D_x u D_x^2 - u D_x^3 - D_x^3 u, \tag{42}$$

and taking the adjoint yields

$$\ell^*[u] = -D_t + D_{xxt} - 3u D_x + 2D_x^2 u D_x - 2D_x u D_x^2 + u D_x^3 \neq \ell[u]. \tag{43}$$

This leads to the adjoint system of the linearized CH equation

$$\ell^*[u]\Lambda = -D_t\Lambda + D_{xxt}\Lambda - 3u D_x\Lambda + 2D_x^2 u D_x\Lambda - 2D_x u D_x^2\Lambda + u D_x^3\Lambda = 0, \tag{44}$$

where Λ is a scalar field. System (44) has solutions $\Lambda = h(x, t)$ given by

$$-D_t h + D_{xxt} h - 3u D_x h + 2D_x^2 u D_x h - 2D_x u D_x^2 h + u D_x^3 h = 0, \tag{45}$$

with no dependence on u. These solutions trivially satisfy the adjoint invariance condition (5) since $\ell^*[U]\Lambda$ and $\partial\Lambda/\partial U,\ \partial\Lambda/\partial D_t U,\ \partial\Lambda/\partial D_x U$, etc., identically vanish for an arbitrary $U(x, t)$. There are no solutions which have explicit local dependence on u or derivatives of u. Evaluating the conservation law formula (7) for $\Lambda = h(x, t)$, we obtain the conserved density

$$\Phi^i[u] = \int_0^1 \frac{d\lambda}{\lambda}\Lambda[U]U|_{U=\lambda u} = \int_0^1 hu d\lambda = hu, \tag{46}$$

where $h(x, t)$ is an arbitrary solution of the adjoint CH equation (45). This yield the infinite number of elementary local conserved densities of the CH equation (41).

2.5. The Generalized Burgers' Equation

Consider the generalized Burgers' equation

$$G[u] = D_t u + u^n D_x u + D_x^2 u = 0, \tag{47}$$

with $n > 0$. This is a first order evolution PDE which has no variational principle directly in terms of u and which reduces for $n = 1$ to ordinary Burgers' equation. Linearizing (47) leads to the operator

$$\ell[u] = D_t + nu^{n-1} D_x u + u^n D_x + D_x^2, \tag{48}$$

where $D_t = \partial_t + u_t\partial_u + u_{tx}\partial_{u_x} + u_{tt}\partial_{u_t} + \cdots$ and $D_x = \partial_x + u_x\partial_u + u_{xx}\partial_{u_x} + u_{tx}\partial_{u_t} + \cdots$ are total derivative operators with respect to t and x. The adjoint yields

$$\ell^*[u] = -D_t - u^n D_x + D_x^2 \neq \ell[u]. \tag{49}$$

The generalized Burgers' equation (47) itself has the form of a local conservation law

$$D_t u + D_x\left(\frac{1}{n+1}u^{n+1} + D_x u\right) = 0. \tag{50}$$

We now consider, more generally, local conservation laws

$$D_t\Phi^0 + D_x\Phi^1 = 0, \tag{51}$$

on all solutions $u(x,t)$ of Eq. (47). Clearly, we are free without loss of generality to eliminate any dependence on u_t (and differential consequences) in the conserved densities Φ^0, Φ^1. All nontrivial conserved densities in this form can be constructed from multipliers Λ on the generalized Burgers' equation, analogous to integrating factors, where Λ depends only on t, x, u and x derivatives of u. In particular, by moving off the generalized Burgers' equation solution space, we have

$$D_t\Phi^0 + D_x\Phi^1 = (u_t + u^n u_x + u_{xx})\Lambda_0 + D_x(u_t + u^n u_x + u_{xx})\Lambda_1 + \cdots, \tag{52}$$

for some expressions $\Lambda_0, \Lambda_1, \cdots$ with no dependence on u_t and differential consequences. This yields (after integration by parts) the multiplier

$$D_t\Phi^0 + D_x(\Phi^1 - \Gamma) = (u_t + u^n u_x + u_{xx})\Lambda, \quad \Lambda = \Lambda_0 - D_x\Lambda_1 + \cdots, \tag{53}$$

where $\Gamma = 0$ when u is restricted to be a generalized Burgers' solution. We now derive an augmented adjoint symmetry determining system which completely characterizes all multipliers Λ.

The definition for multipliers $\Lambda(t, x, u, \partial_x u, \cdots, \partial_x^p u)$ is that $(u_t + u^n u_x + u_{xx})\Lambda$ must be a divergence expression for all functions $u(x,t)$ —not just generalized Burgers' solutions. This determining condition is expressed by

$$\begin{aligned} E_u\big[\big(u_t + u^n u_x + u_{xx}\big)\Lambda\big] = \big(-D_t\Lambda - u^n D_x\Lambda - D_x^2\Lambda\big) + (u_t + u^n u_x + u_{xx})\Lambda_u \\ -D_x[(u_t + u^n u_x + u_{xx})\Lambda_{u_x}] + \cdots \\ +(-1)^p D_x^p\big[\big(u_t + u^n u_x + u_{xx}\big)\Lambda_{\partial_x^p u}\big] = 0, \end{aligned} \tag{54}$$

where $E_u = \partial_u - D_t\partial_{u_t} - D_x\partial_{u_x} + D_t D_x\partial_{u_{tx}} + D_x^2\partial_{u_{xx}} + \cdots$ is the standard Euler operator which annihilates divergence expressions. Equation (54) is linear in $u_t, u_{tx}, u_{txx}, \ldots$, and thus coefficients of u_t and x derivatives of u_t up to order p give rise to a split system of determining equations for Λ. The system is found to consist of the adjoint symmetry determining equation on Λ,

$$-\bar{D}_t - u^n D_x\Lambda - D_x^2\Lambda = 0, \tag{55}$$

and extra determining equations on Λ,

$$\sum_{k=1}^{p}(-D_x)^k \Lambda_{\partial_x^p u} = 0,$$

$$[1-(-1)^q]\Lambda_{\partial_x^q u} + \sum_{k=q+1}^{p} \frac{k!}{q!(k-q)!}(-D_x)^{k-q}\Lambda_{\partial_x^k u} = 0,$$

$$q = 1, \ldots, p-1 \tag{56}$$

$$[1-(-1)^p]\Lambda_{\partial_x^p u} = 0.$$

Here, $\bar{D}_t = \partial_t - (u^n u_x + u_{xx})\partial_u - D_x(u^n u_x + u_{xx})\partial_{u_x} + \cdots$ is the total derivative operator which expresses t derivatives of u through the generalized Burgers' equation (47). Consequently, one is able to work on the space of generalized Burgers' solutions $u(x, t)$ in order to solve the determining system (54) and (55) to find $\Lambda(t, x, u, \partial_x u, \ldots, \partial_x^p u)$. The determining system solutions are the multipliers that yield all nontrivial generalized Burgers' equation conservation laws.

The explicit relation between multipliers Λ and conserved densities Φ^0, Φ^1 for generalized Burgers' conservation laws is summarized as follows.

Given a conserved density Φ^0, we find that a direct calculation of u_t terms in $D_t\Phi^0 + D_x\Phi^1$ yields multiplier equation

$$D_t\Phi^0 + D_x\Phi^1 = (u_t + u^n u_x + u_{xx})\bar{E}_u(\Phi^0) + D_x\Gamma, \tag{57}$$

where $\bar{E}_u = \partial_u - D_x\partial_{u_x} + D_x^2\partial_{u_{xx}} + \cdots$ is a restricted Euler operator and Γ is proportional to $(u_t + u^n u_x + u_{xx})$ and differential consequences. Thus, we obtain the multiplier

$$\Lambda = \bar{E}_u(\Phi^0). \tag{58}$$

Conversely, given a multiplier Λ, we can invert the relation (48) by a standard method (Olver, 1986) using Eq. (57) to obtain the conserved density

$$\Phi^0 = \int_0^1 d\lambda u \Lambda(t, x, \lambda u, \lambda u_x, \lambda u_{xx}, \ldots). \tag{59}$$

From Eqs. (58) and (59), it is natural to define the order of a generalized Burgers' conservation laws as the order of the highest x derivative of u in its multiplier. Thus, we see that all nontrivial generalized Burgers' conservation laws up to order p are determined by multipliers of order p which are obtained as solutions of the augmented system of adjoint symmetry determining Eqs. (55) and (56).

Through the determining system (55) and (56), we now derive a complete classification of all conservation laws (51) up to second order, corresponding to multipliers of the form

$$\Lambda(t, x, u, u_x). \tag{60}$$

The classification results are summarized as follows.

The generalized Burgers' equation (47) for all $n > 0$ admits the multipliers

$$\Lambda = 1, \quad \Lambda = u, \quad \Lambda = u_x + \frac{1}{n+1}u^{n+1}. \tag{61}$$

The only additional admitted multipliers of the form (60) are given by

$$\Lambda = tu - x, \quad \text{if } n = 1, \tag{62}$$

$$\Lambda = t\left(u_x + \frac{1}{3}u^3\right) - \frac{1}{3}xu, \quad \text{if } n = 2. \tag{63}$$

This classifies all nontrivial conservation laws up to second order for the generalized Burgers' equation for any $n > 0$. The conserved densities for these conservation laws are easily obtained using the construction formula (59) as follows.

For the multipliers as shown in (61), we find

$$\Phi^0 = u, \tag{64}$$

$$\Phi^0 = \frac{1}{2}u^2, \tag{65}$$

$$\Phi^0 = \frac{1}{2}uu_x + \frac{1}{(n+1)(n+2)}u^{n+2}, \text{ respectively.} \tag{66}$$

In physical terms, if we regard u as a wave amplitude as in the ordinary ($n = 1$) Burgers' equation, then these conserved densities represent mass, momentum and energy. For the additional multipliers as shown in (62) and (63), we find

$$\Phi^0 = \frac{1}{2}tu^2 - xu, \quad \text{if } n = 1, \tag{67}$$

$$\Phi^0 = \frac{1}{2}tuu_x + \frac{1}{12}tu^4 - \frac{1}{6}xu^2, \quad \text{if } n = 2. \tag{68}$$

3. TRAVELING WAVE SOLUTIONS FOR SOME NLPDES

Now we describe the hyperbolic function method (Bai, 2001; Khater *et al.*, 2001a, 2004b,c; Malfliet, 1991, 1992, 1993, 2004; Malfliet and Hereman, 1996a,b), for a given NLPDE, say, in two variables,

$$f(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \ldots) = 0. \tag{69}$$

We first consider it traveling wave solutions

$$u(x, t) = \phi(\rho), \quad \rho = \eta(x - kt + c), \tag{70}$$

where k, η are constant parameters to be determined, and c is an arbitrary constant, then Eq. (69) becomes an ODE, which is integrated as long as all terms contain derivatives. The associated integration constants can be taken as zero. The next crucial step is that the solution we are looking for is expressed in the form

$$\phi(\rho) = \sum_{i=1}^{n} \sinh^{i-1} \psi (B_i \sinh \psi + A_i \cosh \psi) + A_0, \tag{71}$$

and

$$\frac{d\psi}{d\rho} = \sinh \psi \quad \text{or} \quad \frac{d\psi}{d\rho} = \cosh \psi, \tag{72}$$

where $A_0, \ldots, A_n$ and $B_0, \ldots, B_n$ are constants to be determined. The parameter n will be found by balancing the highest-order nonlinear terms with the highest-order partial derivative term in the given equation and then give the formal solution. Substituting the formal solution (71) and transformation (72) into the ODE obtained above and the change it into hyperbolic polynomial identities for the intermediate variable ψ. Collect all terms with the same power in $\sinh^k \psi \cosh^j \psi$ and set to zero their coefficients to get algebraic relations among the unknowns $k, \eta, A_0, A_1, \ldots, A_n; B_0, B_1 \ldots, B_n$. With the aid of Mathematica and using the Wu's elimination method (Wu, 1994, 1999), we solve for the above unknowns of the equations to finally obtain the traveling wave solutions of the given NLPDE.

3.1. Compound Korteweg-de-Vries (KdV)–Burgers' Equation

For the compound KdV–Burgers' equation (11), which contains some particular important equations such as the compound KdV, KdV–Burgers' and mKdV equations. In order to get a new traveling wave solutions of Eq. (11), substituting (70) into (11), we obtain an ODE

$$k\phi' - p\phi\phi' - q\phi^2\phi' - \eta r\phi'' + s\eta^2\phi''' = 0. \tag{73}$$

According to the hyperbolic function method, it is easy to show that $n = 1$ if balancing ϕ''' with ϕ^2. So we may choose

$$\phi(\rho) = B_1 \sinh \psi + A_1 \cosh \psi + A_0, \tag{74}$$

and target equation

$$\frac{d\psi}{d\rho} = \sinh \psi. \tag{75}$$

With the aid of Mathematica or Maple, from (74) and (75), we can get

$$k\phi' - p\phi\phi' - q\phi^2\phi' - \eta r\phi'' + s\eta^2\phi'''$$
$$= \left\{6A_1 s\eta^2 - q\left(A_1^3 + 3B_1^2 A_1\right)\right\} \sinh^4 \psi + \left(6B_1 s\eta^2 - qB_1^3 - qB_1 A_1^2\right)$$

$$
\begin{aligned}
&\times \sinh^3\psi\cosh\psi + \left(-2pA_1B_1 - 4qA_0B_1A_1 - 2\eta rB_1\right)\\
&\times \sinh^3\psi + \left\{-2r\eta A_1 - p(B_1^2 + A_1^2) - q\left(2A_0B_1^2 + 2A_0A_1^2\right)\right\}\\
&\times \sinh^2\psi\cosh\psi + \left\{-pA_0B_1 + 4A_1s\eta^2 + kA_1 - q\left(A_1A_0^2 + A_1^3\right.\right.\\
&\left.\left.+\,2B_1^2A_1\right)\right\}\sinh^2\psi + \left(s\eta^2B_1 + kB_1 - pA_0B_1 - qB_1A_0^2\right)\sinh\psi\cosh\psi\\
&+(-pB_1A_1 - r\eta B_1 - 2qA_0A_1B_1)\sinh\psi = 0.
\end{aligned}
$$

Setting the coefficients of $\sinh^j\psi\cosh^i\psi$ for $i = 0, 1$ and $j = 1, 2, 3, 4$ to zero, we have the following set of over determined equations in the unknowns A_0, A_1, B_1, η and k :

$$
\begin{aligned}
&6A_1s\eta^2 - q\left(A_1^3 + 3B_1^2A_1\right) = 0,\\
&6B_1s\eta^2 - qB_1^3 - qB_1A_1^2 = 0,\\
&-2pA_1B_1 - 4qA_0B_1A_1 - 2\eta rB_1 = 0,\\
&-2r\eta A_1 - p\left(B_1^2 + A_1^2\right) - q\left(2A_0B_1^2 + 2A_0A_1^2\right) = 0,\\
&-pA_0B_1 + 4A_1s\eta^2 + kA_1 - q\left(A_1A_0^2 + A_1^3 + 2B_1^2A_1\right) = 0,\\
&s\eta^2B_1 + kB_1 - pA_0B_1 - qB_1A_0^2 = 0,\\
&-pB_1A_1 - r\eta B_1 - 2qA_0A_1B_1 = 0.
\end{aligned}
$$

We now solve the above set of equations by using the Wu's elimination method, and obtain the following solutions that include three different cases:
Case 1.

$$
B_1 = 0,\quad A_1 = \pm\sqrt{\frac{6s}{q}}\eta,\quad A_0 = \pm\sqrt{\frac{k - 2s\eta^2}{q}},
$$

$$
k = 12s\eta + r^2 \pm \frac{1}{2}rp\sqrt{\frac{6s}{q}} + \frac{sp^2}{4q},\ sq > 0,\ q(k - 2s\eta^2) > 0.
$$

Case 2.

$$
A_1 = 0,\quad B_1 = \pm\sqrt{\frac{6s}{q}},\quad A_0 = \pm\frac{r}{\sqrt{6sq}},
$$

$$
k = \pm\frac{2pr}{\sqrt{6sq}} + \frac{r^2 + 36s^2}{6s} - 4s\eta^2,\ sq > 0.
$$

Case 3.

$$A_1 = i\sqrt{\frac{3s}{2q}}\eta, \quad B_1 = \pm\sqrt{\frac{3s}{2q}}\eta, \quad A_0 = \pm\frac{r}{\sqrt{6sq}},$$

$$k = \pm\frac{pr}{\sqrt{6sq}} + \frac{r^2}{6s} - 4s\eta^2, \ sq > 0.$$

Now, we consider Eq. (75). By using the separation of variables method the solutions of Eq. (75) are easily written in the following form:

$$\sinh\psi = -\text{cosech}\rho, \quad \cosh\psi = -\coth rho, \tag{76}$$

where set integration constant to zero. Finally, combining (70), (74)–(76) along with Cases 1–3, we find the following three types of traveling wave solutions for the compound KdV–Burgers' equation (11):
Type 1.

$$u_1(x,t) = \pm\sqrt{\frac{k-2s\eta^2}{q}} \mp \sqrt{\frac{6s}{q}}\eta\coth\eta(x-kt+c). \tag{77}$$

Type 2.

$$u_2(x,t) = \pm\frac{r}{\sqrt{6sq}} \mp \sqrt{\frac{6s}{q}}\,\text{cosech}\,\eta(x-kt+c). \tag{78}$$

Type 3.

$$u_3(x,t) = \pm\frac{r}{\sqrt{6sq}} + i\sqrt{\frac{3s}{2q}}\eta\coth\eta(x-kt+c)$$
$$\pm\sqrt{\frac{3s}{2q}}\eta\,\text{cosech}\,\eta(x-kt+c). \tag{79}$$

By means of the same procedures described above, we obtain solutions of (17), (18) and (19):

(1) For the compound KdV as in (17), we obtain the following three types of solitary wave solutions
Type 1.

$$u_4(x,t) = \pm\sqrt{\frac{k-2s\eta^2}{q}} \mp \sqrt{\frac{6s}{q}}\eta\coth\eta(x-kt+c), \tag{80}$$

where

$$k = 12s\eta + \frac{sp^2}{4q}, \quad sq > 0,\ q(k - 2s\eta^2) > 0.$$

Type 2.

$$u_5(x, t) = \pm\sqrt{\frac{6s}{q}}\,\mathrm{cosech}\;\eta(x - kt + c), \ \ k = 6s - 4s\eta^2,\ sq > 0. \quad (81)$$

Type 3.

$$u_6(x, t) = i\sqrt{\frac{3s}{2q}}\eta\coth\eta(x - kt + c) \pm \sqrt{\frac{3s}{2q}}\eta\,\mathrm{cosech}\eta(x - kt + c),$$
$$k = -4s\eta^2,\ sq > 0. \quad (82)$$

(2) For the KdV–Burgers' equation (18), we obtain the following three types of solitary wave solutions
Type 1.

$$u_7(x, t) = \pm\sqrt{\frac{k - 2s\eta^2}{q}} \mp \sqrt{\frac{6s}{q}}\eta\coth\eta(x - kt + c), \quad (83)$$

where

$$k = 12s\eta + r^2,\ sq > 0,\ q(k - 2s\eta^2) > 0.$$

Type 2.

$$u_8(x, t) = \pm\frac{r}{\sqrt{6sq}} \mp \sqrt{\frac{6s}{q}}\,\mathrm{cosech}\eta(x - kt + c), \quad (84)$$

where

$$k = \frac{r^2 + 36s^2}{6s} - 4s\eta^2,\ sq > 0.$$

Type 3.

$$u_9(x, t) = \pm\frac{r}{\sqrt{6sq}} + i\sqrt{\frac{3s}{2q}}\eta\coth\eta(x - kt + c)$$
$$\pm\sqrt{\frac{3s}{2q}}\eta\,\mathrm{cosech}\;\eta(x - kt + c), \quad (85)$$

where

$$k = \frac{r^2}{6s} - 4s\eta^2,\ sq > 0.$$

(3) For the mKdV as in (19), we obtain the following three types of solitary wave solutions

Type 1.

$$u_{10}(x,t) = \pm\sqrt{\frac{k-2s\eta^2}{q}} \mp \sqrt{\frac{6s}{q}}\eta \coth \eta(x-kt+c), \tag{86}$$

where

$$k = 12s\eta,\ sq > 0,\ q(k-2s\eta^2) > 0.$$

Type 2.

$$u_{11}(x,t) = \pm\sqrt{\frac{6s}{q}} \operatorname{cosech} \eta(x-kt+c), \quad k = 6s - 4s\eta^2,\ sq > 0. \tag{87}$$

Type 3.

$$u_{12}(x,t) = i\sqrt{\frac{3s}{2q}}\eta \coth \eta(x-kt+c) \pm \sqrt{\frac{3s}{2q}}\eta \operatorname{cosech} \eta(x-kt+c),\ \ k = -4s\eta^2,\ sq > 0. \tag{88}$$

3.2. Liouville's Equation

Now, we generate a new soliton solution class for Liouville's equation by using the Bäcklund Transformations (BTs). Here, we use this transformation for finding a solution of Liouville's equation (20).

Let

$$T = \frac{1}{2}(ix+t), \qquad X = \frac{1}{2}(ix-t), \tag{89}$$

we obtain a form of Liouville's equation

$$u_{TX} = -e^u. \tag{90}$$

It is known that Liouville's equation can be derived from the following Ablowitz–Kaup–Newell–Segur (AKNS) system

$$\phi_X = P\phi, \quad \phi_T = Q\phi, \tag{91}$$

where

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \tag{92}$$

P and Q are two 2×2 null-trace matrices.

$$P = \begin{pmatrix} \frac{\eta}{2} & q \\ r & -\frac{\eta}{2} \end{pmatrix}, \quad Q = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}, \tag{93}$$

where

$$r = q = \frac{U_X}{2}, \qquad A = B = -C = \frac{-e^u}{2\eta}.$$

Here, η is a parameter, where P and Q must satisfy the following integrability condition:

$$P_T - Q_X + PQ - QP = 0. \tag{94}$$

Konno and Wadati (1975) introduced the function

$$\Gamma = \frac{\phi_1}{\phi_2}, \tag{95}$$

and, for Liouville's equation (90), derived a BT with the following form

$$u' = -u \pm 4 \tanh^{-1} \Gamma. \tag{96}$$

We shall choose a known solution of the above Liouville's equation (96) as a traveling wave $u(X, T)$ and substitute these solution into the corresponding matrices P and Q. Next we solve (95) for ϕ_1 and ϕ_2. Then by (95) and the corresponding BTs (96), we shall obtain the new solution classes. By direct calculation we take

$$u = \ln\left(\frac{2k}{\rho^2}\right), \tag{97}$$

as a traveling wave solution class of the Liouville's equation (90). The traveling wave known solution of the Liouville's equation takes the form

$$u = u(\rho), \quad \rho = X - kT. \tag{98}$$

In this case, the AKNS system (91) and (92) has a general solution. Let us consider the more general case. Suppose that the components q and r of the matrix P are function of ρ (Miura, 1976; Khater *et al.*, 1998):

$$q = q(\rho), \quad r = r(\rho); \tag{99}$$

then the components A, B and C of the matrix Q as determined by (93) are also functions of ρ:

$$A = A(\rho), \quad B = B(\rho), \quad C = C(\rho). \tag{100}$$

Under these assumptions, the following result holds, which is crucial in the subsequent exact solution. The quantity

$$\beta_1 = \left(A + k\frac{\eta}{2}\right)^2 + (B + kq)(C + kr), \tag{101}$$

is constant with respect to ρ (or x and t). Using the result of (Khater *et al.*, 2006) and the constant β_1 defined by (101) is zero and the corresponding solutions of the AKNS system (91) and (92) are:

$$\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} (C + kr)^{-1/2}\left[\left(A + k\frac{\eta}{2}\right)(c_1\xi + c_2) + c_1\right] \\ (C + kr)^{1/2}(c_1\xi + c_2) \end{bmatrix}, \text{ when } \beta_1 = 0, \tag{102}$$

where c_1 and c_2 are constants and

$$\xi = T + \int \frac{r\, d\rho}{C + kr}. \tag{103}$$

Now applying the results obtained here and the known traveling wave solutions for the Liouville's equation respectively to construct new solution class of the corresponding Liouville's equation by means of the BTs. The constant β_1 and ξ defined by (101) and (103) can be determined by using (97)

$$\xi = T + \frac{1}{k}\rho + \frac{1}{\eta k}\ln(\eta\rho - 1). \tag{104}$$

Consequently, we obtain Γ from (102) for $\beta_1 = 0$

$$\Gamma = (C + kr)^{-1}\left[\left(A + k\frac{\eta}{2}\right) + 1/(\xi + c_0)\right], \quad c_0 = c_1/c_2, \tag{105}$$

then substituting this Γ into the BTs (96) and using (97), we arrive at the new solution u' of the Liouville's equation (90) corresponding to the known traveling wave solution class (97), then

$$u' = -\ln\frac{2k}{\rho^2} \pm 4\tanh^{-1}\Gamma, \quad \rho = X - kT, \tag{106}$$

$$\Gamma = \frac{\eta\rho^2}{1 - \eta\rho}\left[\left(\frac{-1}{\eta\rho^2} + \frac{\eta}{2}\right) + 1/\left(\rho + \frac{1}{\eta}\ln(\eta\rho - 1) + k(T + c_0)\right)\right].$$

Then the exact solution of Liouville's equation (20) takes the form

$$u' = -\ln\frac{2k}{\rho^2} \pm 4\tanh^{-1}\Gamma, \quad \rho = \frac{1}{2}[(i - k)x + (k - 1)t]. \tag{107}$$

3.3. SG Equation

Now, we shall find a traveling wave solution class $u'(x, t)$ of the SG equation (32) by means of the BTs.

In the following, we omit some tedious calculations but only list the main results of the SG equation. Proceeding as for Liouville's equation we obtain

$$P = \begin{pmatrix} \frac{\eta}{2} & \frac{-u_x}{2} \\ \frac{u_x}{2} & -\frac{\eta}{2} \end{pmatrix}, \quad Q = \begin{pmatrix} \frac{\cos u}{2\eta} & \frac{-\sin u}{2\eta} \\ \frac{\sin u}{2\eta} & \frac{-\cos u}{2\eta} \end{pmatrix}, \tag{108}$$

$$u' = u + 4\tan^{-1}\Gamma, \tag{109}$$

$$u = 4\tan^{-1}[\exp(\eta\rho)], \quad \rho = x + kt, \quad k = \frac{1}{\eta^2}, \tag{110}$$

$$u' = 4\tan^{-1}\frac{\exp(\eta\rho) + \Gamma}{1 - \Gamma\exp(\eta\rho)}, \tag{111}$$

$$\Gamma = (C + kr)^{-1}\left[\left(A + k\frac{\eta}{2}\right) + 1/(\xi + c_0)\right], \quad c_0 = c_1/c_2, \tag{112}$$

$$\xi = t - \frac{4\ln(2\sinh\eta\rho - \cosh\eta\rho) - 3\ln(\sinh\eta\rho - \cosh\eta\rho) - \eta\rho}{6\eta}. \tag{113}$$

3.4. CH Equation

We seek a solution of CH equation (41),
let

$$u(x,t) = \phi(\rho), \quad \rho = x - kt + c \quad \text{where } c \text{ is a constant} \tag{114}$$

then we obtain the equation

$$-k\phi' + k\phi''' + 3\phi\phi' - 2\phi'\phi'' - \phi\phi''' = 0, \quad \text{where } \phi' = \frac{d\phi}{d\rho}. \tag{115}$$

By using the substitution

$$\phi = A\,\mathrm{cosech}^n\rho, \tag{116}$$

inserting Eq. (116) into (117), we obtain

$$A = k, \quad n = -1,$$

then we obtain a solution class of CH equation (41) as

$$u(x,t) = k\cosh\rho, \quad \rho = x - kt + c. \tag{117}$$

3.5. The Generalized Burgers' Equation

We seek a solution of generalized Burgers' equation (47),

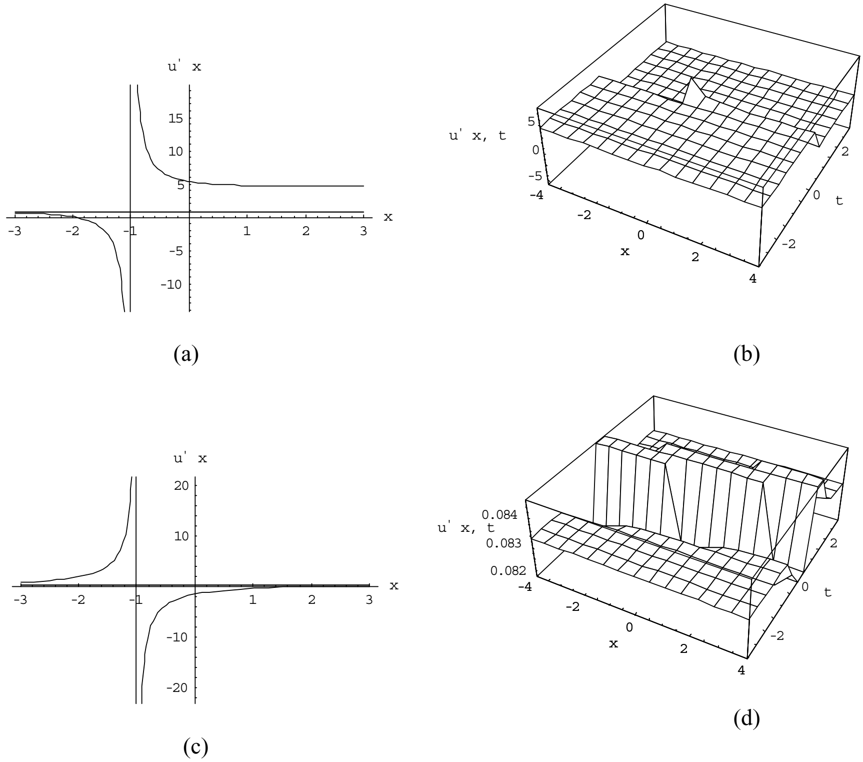

Fig. 1. (a) The single-antikink solution (77) in the plane with $\eta = 1, r = 1, q = 6, s = 4, p = 2$ at different times $t = 0, 1, 2$; (b) The single-antikink solution (77) in the surface with $k = 155/3$ and $\eta = 1$; (c) The single-kink solution (78) in the plane with $\eta = 1, r = 1, q = 6, s = 4, p = 2$ at different times $t = 0, 1, 2$; (d) The single soliton solution (78) in the surface with $k = 201/24$ and $\eta = 1$.

let

$$u(x, t) = \phi(\rho), \quad \rho = \eta(x - kt + c) \quad \text{where } c \text{ is a constant} \tag{118}$$

Equation (47) becomes

$$-k\frac{d\phi}{d\rho} + \phi^n \frac{d\phi}{d\rho} + \eta \frac{d}{d\rho}\left(\frac{d\phi}{d\rho}\right) = 0. \tag{119}$$

We assume that

$$\phi(\rho) \to 0, \frac{d\phi}{d\rho} \quad \text{and} \quad \frac{d^2\phi}{d\rho^2} \to 0 \quad \text{as} \quad \rho \to \pm\infty. \tag{120}$$

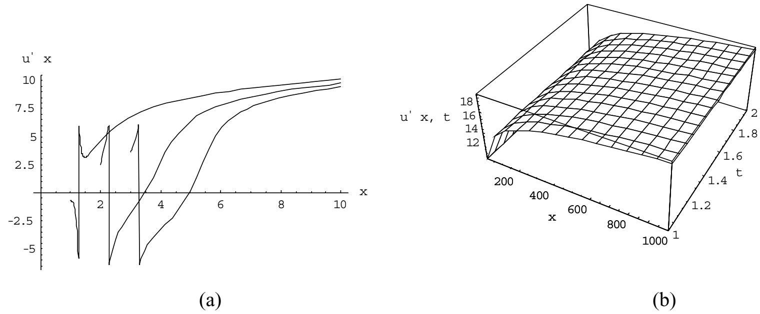

Fig. 2. (a) The single soliton solution (106) in the plane at different times $t = 0, 1, 2$; (b) The single-kink solution (106) in the surface with $k = 1$ and $\eta = 1$.

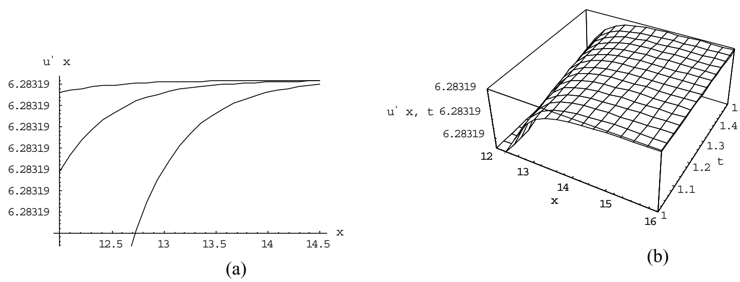

Fig. 3. (a) The single-kink solution (111) in the plane at different times $t = 0, 1, 2$; (b) The single-kink solution (111) in the surface with $k = 1,\ \eta = 1$.

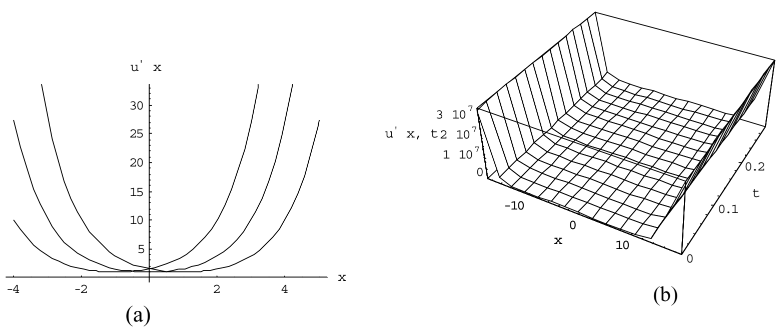

Fig. 4. (a) The solution (117) in the plane at different times $t = 0, 1, 2$; (b) The solution (117) in the surface with $k = 1,\ c = 1$.

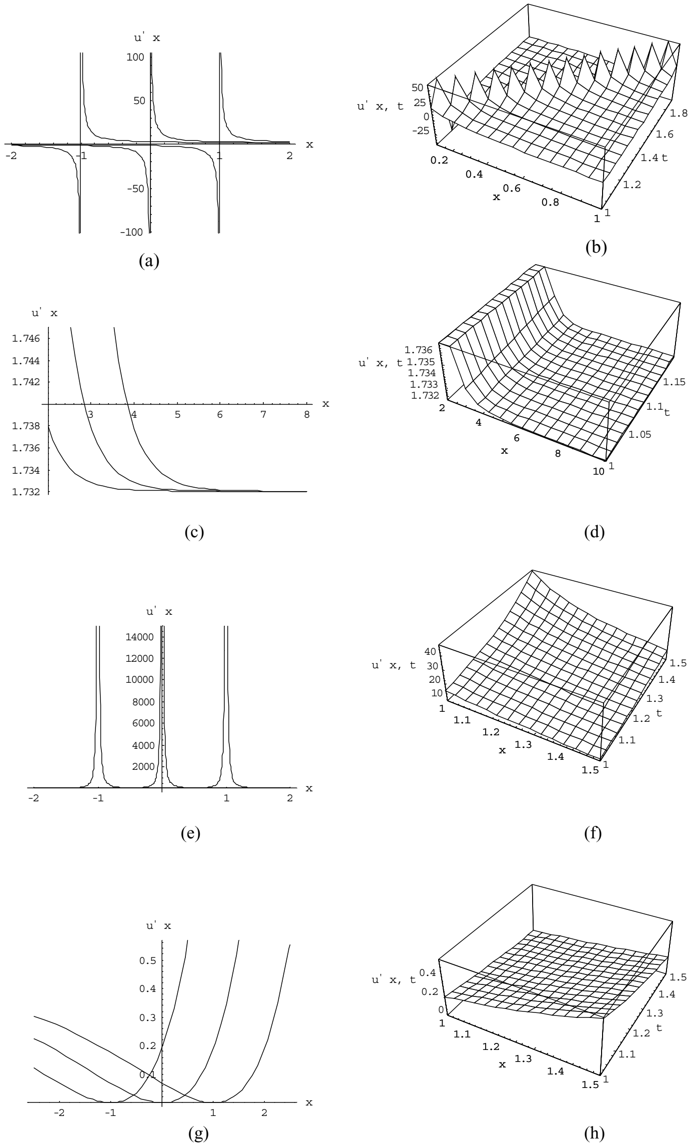

Fig. 5. (a) The solution (122) in the plane with $\eta = 1, k = 1, n = 1$ and different times $t = 0, 1, 2$; (b) The solution (122) in the surface with $\eta = k = 1,\ n = 1$; (c) The solution (122) in the plane with $\eta = 1, k = 1, n = 2$ and different times $t = 0, 1, 2$; (d) The solution (122) in the surface with $n = 2,\ \eta = 1$; (e) The solution (122) in the plane with $\eta = 1, k = 1, n = 1/2$ and different times $t = 0, 1, 2$; (f) The solution (122) in the surface with $\eta = k = 1,\ n = 1/2$; (g) The solution (122) in the plane with $\eta = 1, k = 1, n = -1/2$ and different times $t = 0, 1, 2$; (h) The solution (122) in the surface with $n = -1/2,\ \eta = 1$.

Integrating Eq. (119), we get

$$k\phi - \frac{\phi^{n+1}}{n+1} = \eta \frac{d\phi}{d\rho}, \tag{121}$$

by using the separation of variables and integrating Eq. (121), we obtain

$$\phi = [(n+1)k]^{\frac{1}{n}} \left[1 - A \exp \frac{-kn\rho}{\eta} \right]^{\frac{-1}{n}}, \tag{122}$$

where A is a constant of integration, $\rho = \eta(x - kt + c)$.

4. CONCLUSIONS

In this paper, we considered the construction of conservation laws to some NLPDEs (compound Korteweg–de-Vries-Burgers', Liouville, SG, CH, and the generalized Burgers' equations) by using the effective algorithmic method. Exact traveling wave and solitary wave solutions for compound Korteweg–de-Vries–Burgers' equation (which contains some particular important equations such as the compound KdV, KdV-Burgers', and mKdV equations) are obtained by using hyperbolic function method and the Wu's elimination method. It has been shown that the implementation of certain BTs for a class of NLPDE requires the solution of the underlying linear differential equation whose coefficients depend on the initial known solution $u(x, t)$ of the NLPDE. We obtain traveling wave solutions for Liouville and SG equations by using BTs. On the other hand, other methods and transformations are developed to obtain exact solutions for CH and generalized Burgers' equations.

The solutions (77), (78), (106), and (111) are the single-kink and antikink solutions [see Figs. 1(a–c); 2(a)(b); 3(a) and (b)] corresponding to the eigenvalue η. The single soliton solutions (78) and (106) are characterized by the eigenvalue η [see Figs. (1d) and (2a)]. Other solutions (117) and (122) are characterized by the eigenvalue η [see Figs. (4a)–(5h)].

REFERENCES

Ablowitz, M. J. and Clarkson, P. A. (1991). Solitons, Nonlinear evolution equations and inverse scattering. *London Mathematical Society* Lecture Notes. Cambridge University Press, Cambridge.

Anco, S. C. and Bluman, G. (1996). Derivation of conservation laws from nonlocal symmetries of differential equations. *Journal of Mathematical Physics* **37**, 2361.

Anco, S. C. and Bluman, G. (1997). Direct construction of coservation laws from field equations. *Physical Review Letters* **78**, 2869.

Anco, S. C. and Bluman, G. (1998). Integrating factors and first integrals of ordinary differential equations. *European Journal of Applied Mathematics* **9**, 254.

Anco, S. C. and Bluman, G. (2001). Direct construction method for coservation laws of partial differential equations. Part I: Examples of conservation laws classifications. *European Journal of Applied Mathematics* **13**, 547.

Anco, S. C. and Bluman, G. (2002). Direct construction method for coservation laws of partial differential equations. Part II: General treatment. *European Journal of Applied Mathematics* **13**, 567.

Bai, C. L. (2001). Exact solutions for nonlinear partial differential equation: A new approach. *Physics Letters A* **288**, 191.

Bessel–Hagen, E. (1921). Über die Erhaltungsätze der Elektrodynami. *Mathematiche Annalen* **84**, 258.

Bluman, G. (1990). Simplifying the form of Lie groups admitted by a given differential Equation. *Journal of Mathematical Analysis and Applications* **145**, 52.

Bluman, G. (1993). Use and construction of potential symmetries. *Journal of Mathematical and Computer Modelling* **8**, 1.

Bluman, G. and Kumi, S. (1987). On invariance properties of the wave equation. *Journal of Mathematical Physics* **28**, 207.

Bluman, G. and Kumi, S. (1988). Exact solutions for wave equations of two-layered media with smooth transition. *Journal of Mathematical Physics* **29**, 86.

Bluman, G. and Kumi, S. (1989). Symmetries and differential Equations, Springer, New York.

Bluman, G., Kumi, S., and Reid, G. (1988). New classes of symmetries of partial differential equations. *Journal of Mathematical Physics* **29**, 801.

Cariello, F. and Tabor, M. (1991). Similarity reductions from extended Painlevé expansions for nonintegrable evolution equations. *Physica D* **53**, 59.

Gardner, C. S., Greene, J. M., Kruskal, M. D., and Miura, R. M. (1967). Method for solving the korteweg-de Vries equation. *Physical Review Letter* **19**, 1095.

Hirota, R. (1973). Exact N-soliton solutions of the wave equation of long waves in shallow-water and in nonlinear lattices. *Journal of Mathematical Physics* **14**, 810.

Khater, A. H., Abdalla, A. A., Callebaut, D. K., and Ramady, A. G. (1999). Rational reflection coefficients in inverse scattering for a Dirac system. *Inverse Problem* **15**, 241.

Khater, A. H., Callebaut, D. K., and El-Kalaawy, O. H. (1998). Bäcklund transformations and exact soliton solutions for nonlinear Schrödinger-type equations. *Il Nuovo Cimento B* **113**, 1121.

Khater, A. H., Callebaut, D. K., Ramady, A. G., and Abdul Aziz, S. F. (1997). On the invariant variational principle for Sturm-Liouville equation, linear and nonlinear Dirac system. *Il Nuovo Cimento B* **110**, 1297.

Khater, A. H., Callebaut, D. K., and Sayed, S. M. (2004a). Conservation laws for some nonlinear evolution equations which describe pseudo-spherical surfaces. *Journal of Geometry and Physics* **51**, 332.

Khater, A. H., Callebaut, D. K., and Sayed, S. M. (2006). Exact solutions for some nonlinear evolution equations which describe pseudospherical surfaces. *Journal of Computational and Applied Mathematics* **189**, 387.

Khater, A. H., Malfliet, W., Callebaut, D. K., and Abdul Aziz, S. F. (2004b). On the similarity solutions of the inhomogeneous nonlinear diffusion equation via symmetry method. *Journal of the Physical Society of Japan* **73**, 42.

Khater, A. H., Malfliet, W., and Kamel, E. S. (2002a). Travelling wave solutions of some classes of nonlinear equations in $(1+1)$ and $(2+1)$ dimensions. *Journal of Computational and Applied Mathematics* **140**, 469.

Khater, A. H., Malfliet, W., and Kamel, E. S. (2004c). Travelling wave solutions of some classes of nonlinear equations in $(1+1)$ higher dimensions. *Mathematics and Computers in Simulation* **64**, 247.

Khater, A. H., Moussa, M. H., and abdul Aziz, S. F. (2001a). On the similarity solutions of the variable-coefficient semi-linear hyperbolic equation $u_{xt} = \alpha(t)f(u)$ via a symmetry method. *Il Nuovo Cimento* **B116**, 889.

Khater, A. H., Moussa, M. H., and abdul Aziz, S. F. (2001b). The invariant variational principle and exact solutions for the generalized Sturm-Liouville equation and linear Dirac system, Il Nuovo Cimento **B116**, 1203.

Khater, A. H., Moussa, M. H., and abdul Aziz, S. F. (2002b), Potential symmetries and invariant solutions for the generalized one-dimensional Fokker-Planck equation, *Physica* **A304**, 395.

Khater, A. H., Moussa, M. H., and Abdul Aziz, S. F. (2002c), Potential symmetries and invariant solutions for the inhomogeneous nonlinear diffusion equation. *Physica* **A312**, 99.

Khater, A. H., Moussa, M. H., and abdul Aziz, S. F. (2003). On the similarity solutions of the Calogero-Degasperis-Fokas modified KdV equation via symmetry method. *Journal of the Physical Society of Japan* **72**, 2523.

Konno, K. and Wadati, M. (1975). Simple derivation of Bäcklund Transformation from Riccati form of inverse method. *Progressive Theoretical Physics* **53**, 1652.

Malfliet, W. (1991). Series solution of nonlinear coupled reaction-diffusion equations. *Journal of Physics A: Mathematical and general* **24**, 5499.

Malfliet, W. (1992). Solitary wave solutions of nonlinear wave equations. *American Journal of Physics* **60**, 650.

Malfliet, W. (1993). Approximate solution of the damped Burgers' equation.*Journal of Physics A: Mathematical and general* **26**, 723.

Malfliet, W. (2004). The Tanh method: a tool for solving certain classes of nonlinear evolution and wave equations. *Journal of Computational and Applied Mathematics* **164**, 529.

Malfliet, W. and Hereman, W. (1996a), The Tanh method. Part I. Exact solutions of nonlinear evolution and wave equations. *Physica Scripta* **54**, 563.

Malfliet, W. and Hereman, W. (1996b). The Tanh method. Part II. Perturbation technique for conservative systems. *Physica Scripta* **54**, 569.

Mikhailov, A. V. (1981). The reduction problem and the inverse scattering method. *Physica* **D3**, 73.

Miura, R. M. (1976). Bäcklund transformations, the inverse scattering method. Solitons and their applications Lecture Notes in Mathematics, *vol*. 515, Springer-Verlag, New York.

Miura, R. M., Gardner, C. S., and Kruskal, M. D. (1968). Korteweg – de Vries equations and generalizations: II. Existence of conservation laws and constants of motion. *Journal of Mathematical Physics* **9**, 1204.

Noether, E. (1918). Invariante Variations problem, Nachr. König. Gesell. Wissen. Göttingen, Math. Phys. **K1.**, 235.

Olver, P. (1977). Evolution equations possessing infinitely many symmetries. *Journal of Mathematical Physics* **18**, 1212.

Olver, P. (1986). Applications of Lie Groups to Differential Equations, Springer-Verlag, New York.

Otwinowski, M., Paul, R., and Laidlaw, W. G. (1988). Exact travelling wave solutions of a class of nonlinear diffusion equations by reduction to a quadrature. *Physics letters* **A128**, 483.

Tonti, E. (1967). Variational principle in elastostatics. *Meccanica* **2**, 201.

Tonti, E. (1968a). Variational principles in electromagnetism. *Rendiconti Istituto Lombardo di Scienze a Lettere* **102**, 845.

Tonti, E. (1968b). Variational formulation for linear equations of mathematical physics. *Rend. Acc. Naz. Lincei* Serie VIII, *vol*. **XLIV**, 75.

Tonti, E. (1968c). Gauge invariance and conservation laws. *Rend. Accad. Naz. Lincei vol*. **VL**, 293.

Tonti, E. (1984). Variational formulation for every nonlinear problem. *International Journal of Engineering Science* **22**, 1343.

Wang, M. (1995). Solitary wave solutions for variant Boussinesq equations. *Physics letters* **A199**, 169.

Wang, M. (1996). Exact solution for a compound KdV–Burgers' equation. *Physics letters* **A213**, 279.

Weiss, J., Tabor, M., and Carnevale, G. (1983). The Painlevé property for partial differential equations. *Journal of Mathematical Physics* **24**, 522.
Whitham, G. B. (1974). *Linear and Nonlinear Waves* Wiley, New York.
Wolf, T. (2002). A comparison of four approaches to the calculation of conservation laws. *European Journal of Applied Mathematics* **13**, 129.
Wu, W. T. (1999). Polynomial equations-solving and its applications, Algorithms and computation, 1–9, Lecture Notes in Computer Science. *vol*.834, Springer-Verlag, Berlin, 1994.